

A Solvable Two-Charge Ensemble on the Circle

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Abstract

We introduce an ensemble consisting of logarithmically repelling charge one and charge two particles on the unit circle constrained so that the total charge of all particles equals N , but the proportion of each species of particle is allowed to vary according to a fugacity parameter. We identify the proper scaling of the fugacity with N so that the proportion of each particle stays positive in the $N \rightarrow \infty$ limit. This ensemble forms a Pfaffian point process on the unit circle, and we derive the scaling limits of the matrix kernel(s) as a function of the interpolating parameter. This provides a solvable interpolation between the circular unitary and symplectic ensembles.

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At their core, classical random matrix ensembles consist of identical jointly distributed random variables which demonstrate repulsion. These random variables are often identified with eigenvalues of matrices chosen with respect to some probability measure (for instance on the entries of the matrices) or with interacting charged particles in the complex plane in the presence of a potential, which constrains the particles to a bounded region. Three of the simplest examples of such ensembles are Dyson's circular orthogonal/unitary/symplectic ensembles $C(O/U/S)E$, which give rise to random variables with joint density defined on the N -fold copy of the unit circle, \mathbb{T}^N , by

$$\Omega_{N,\beta}(\zeta) = \frac{1}{Z_{N,\beta}} \prod_{m < n}^N |\zeta_n - \zeta_m|^\beta,$$

where $\beta = 1$ for COE, $\beta = 2$ for CUE and $\beta = 4$ for CSE, and $Z_{N,\beta}$ is a normalizing constant, the *partition function* responsible for ensuring that Ω_N defines a probability density.

These specific values of the parameter β lead from well-understood algebraic identities to the expression of the correlation functions (or joint intensities, defined below) in terms of determinants (when $\beta = 2$) or Pfaffians (when $\beta = 1, 4$) of matrices whose entries are determined by a kernel (of the reproducing sort) dependent on β and N . Scaling limits of these kernels as $N \rightarrow \infty$ then tell us about local interactions between the random variables as their numbers increase to infinity.

The special determinantal/Pfaffian structure does not seem to exist for other values of β (see for instance, however [11]), so that while it is perfectly reasonable to generalize the joint density $\Omega_{N,\beta}$ to other values of β , and hence to interpolate between the classical ensembles,

the determinantal/Pfaffian expression for the correlation functions does not persist along the entirety of this interpolation. The goal here is to demonstrate a different interpolation between COE and CSE, which has Pfaffian intensities for all values of the interpolating parameter and for which we can explicitly compute the kernel (as a function of N and the parameter) and its scaling limits as $N \rightarrow \infty$.

1 The Two-Charge Model

The model here is the circular version of the two-charge model introduced in [9]. We will cover the basics here, but refer the reader to that article for a more in-depth discussion. The reader new to the connection between random matrix theory and two-dimensional electrostatics should also consult Forrester's book [4].

We suppose we have L charge one particles and M charge two particles constrained to the unit circle \mathbb{T} and interacting logarithmically, so that the interaction between a particle of charge q_1 located at ζ_1 and a particle of charge q_2 located at ζ_2 is given by $-q_1 q_2 \log |z_1 - z_2|$. Thus, if the L charge one particles are located at $\xi_1, \xi_2, \dots, \xi_L$ and the M charge two particles are at $\zeta_1, \zeta_2, \dots, \zeta_M$, the total interaction energy of the system is given by

$$E_{L,M}(\xi, \zeta) := - \sum_{\ell < k} \log |\xi_k - \xi_\ell| - 4 \sum_{m < n} \log |\zeta_n - \zeta_m| - 2 \sum_{m=1}^M \sum_{\ell=1}^L \log |\zeta_m - \xi_\ell|.$$

When the system is at fixed temperature T , the probability (density) of finding the system in the state determined by (ξ, ζ) is given by

$$\Omega_{L,M}(\xi, \zeta) = \frac{1}{L!M!Z_{L,M}} e^{-\frac{1}{kT} E_{L,M}(\xi, \zeta)},$$

where $Z_{L,M}$ is a normalizing constant and k is a constant with units so that $b := (kT)^{-1}$ is a unit-less temperature parameter¹. We will assume throughout that $b = 1$.

We now suppose the number of each species are random variables constrained so that the sum of all charges is N , and the probability of the system having L charge one and M charge two particles is given by

$$X^L \frac{Z_{L,M}}{Z_N(X)}$$

for positive *fugacity* parameter X , where

$$Z_N(X) = \sum_{L+2M=N} \frac{X^L}{L!M!} \int_{\mathbb{T}^L} \int_{\mathbb{T}^M} e^{-b E_{L,M}(\xi, \zeta)} d\mu^L(\xi) d\mu^M(\zeta),$$

and μ and μ^L are Lebesgue (Haar) measure² on \mathbb{T} and \mathbb{T}^L . Under this paradigm, our system consists of two species of particles with total charge summing to N , and X being a parameter which controls the proportion of each species of particle.

¹This parameter is usually denoted β , but that symbol is already taken by the parameter interpolating between COE/CUE/CSE.

²The normalization of Haar measure is unimportant as long as it is done consistently. Here we will take $\mu(\mathbb{T}) = 2\pi$.

Theorem 1.1. *If N is even*

$$Z_N(X) = (2\pi)^{\lfloor \frac{N+1}{2} \rfloor} \prod_{n=1}^{\lfloor \frac{N}{2} \rfloor} \frac{(2X)^2 + (N - 2n + 1)^2}{N - 2n + 1},$$

and if N is odd,

$$Z_N(X) = (2\pi)^{\lfloor \frac{N+1}{2} \rfloor} X \prod_{n=1}^{\lfloor \frac{N}{2} \rfloor} \frac{(2X)^2 + (N - 2n + 1)^2}{N - 2n + 1}.$$

We will often have occasion to compare various particle statistics of our system with the two-charge model considered in [9]. In that model (which we will call the RSX model), the charged particles are restricted to the line (identified with \mathbb{R}) and in the presence of the harmonic oscillator potential—a potential which keeps the repelling particles from fleeing to infinity. The authors of that work concentrated on global statistics: the expected number/proportion of each species of particles and the spatial density for various scalings of the fugacity in the large N limit. In our case, the limiting expected number/proportion of particles is certainly of interest (not the least because it requires a different scaling of the fugacity in order to ensure a positive proportion of each species of particle). On the other hand, the spatial density in our circular model is trivial, since symmetry demands each species be uniformly distributed on \mathbb{T} , independent of N .

The analogy between the circular two charge model here and the RSX model extends the analogy between the Gaussian Hermitian ensembles, G(O/S)E and C(O/S)E. And, as it was for Dyson, the extreme symmetry present in the circular two-charge ensemble simplifies the derivation of certain quantities of interest. For instance, the symmetry of the circle will allow us to solve for the fluctuations³ in spacings between particles; this problem is still unsolved in the RSX model, though universality, suggests that the fluctuations/local statistics of particles reported here will be the same as in the RSX model. Moreover, when the fugacity suitably scales with N we see an explicit interpolation of matrix kernels between the limiting bulk kernels for $\beta = 1$ (*i.e.* GOE, COE) and $\beta = 4$ (GSE, CSE) ensembles.

2 Global Statistics

If we denote by $L_N(X)$ the random variable giving the number of charge 1 particles when the fugacity is X and the total charge of the system is N , then $Z_N(TX)/Z_N(X)$, as a function of T is the probability generating function for $L_N(X)$. This allows us to recover the limiting (large N) probability generating function for the number of charge one particles as a function of X . Since the number of charge 1 particles has the same parity as N , we expect different answers for N even and odd. Taking this into account, one sees that in the limit, the number of charge 1 particles is essentially Poisson.

Corollary 2.1. *The limiting probability generating function for the number of charge 1 particles is, when N is even,*

$$\lim_{N \rightarrow \infty} \frac{Z_N(TX)}{Z_N(X)} = \frac{\cosh(\pi XT)}{\cosh(\pi X)},$$

³Those which follow from the scaling limits for all matrix kernels.

and when N is odd,

$$\lim_{N \rightarrow \infty} \frac{Z_N(TX)}{Z_N(X)} = \frac{\sinh(\pi XT)}{\sinh(\pi X)}.$$

From this limiting probability generating function we see that, in the limit, the expected number of charge 1 particles is finite, though the exact expectation is dependent on X , and (restricting ourselves to the even case for the moment) is explicitly given by $\pi X \tanh(\pi X)$. Thus, if we are interested in a limiting situation with a non-trivial proportion of charge 1 particles, we need to scale X with N . This calculation suggests the proper scaling of X necessary to achieve this goal is linear; that is $X = Nr$.

Theorem 2.2. For $r > 0$,

$$\lim_{N \rightarrow \infty} \frac{1}{N} E[L_N(Nr)] = 2r \arctan\left(\frac{1}{2r}\right),$$

and

$$\lim_{N \rightarrow \infty} \frac{1}{N} \text{var}(L_N(Nr)) = 2r \arctan\left(\frac{1}{2r}\right) - \frac{4r^2}{1 + 4r^2}.$$

It is interesting to note that the scaling necessary to simultaneously achieve a non-trivial proportion of each species is different here than that for the RSX model. In the RSX case, the fugacity must scale with \sqrt{N} to achieve this sort of equilibrium.

Unsurprisingly, we find a Central Limit Theorem for the number of charge 1 particles.

Theorem 2.3. For $r > 0$, set

$$\mu_N = 2Nr \arctan\left(\frac{1}{2r}\right)$$

and

$$\sigma_N^2 = N \left(2r \arctan\left(\frac{1}{2r}\right) - \frac{4r^2}{1 + 4r^2} \right),$$

then

$$\frac{L_N(Nr) - \mu_N}{\sigma_N}$$

converges in distribution to a standard normal random variable.

3 Local Statistics

Given a measurable set $A \subseteq \mathbb{T}$ we define the integer valued random variables $N_A^{(1)}$ and $N_A^{(2)}$ to be respectively the number of charge 1 and charge 2 particles in A . Of course $N_A^{(1)}$ and $N_A^{(2)}$ depend also on X , but we leave that dependence implicit. If for collections of mutually disjoint sets A_1, A_2, \dots, A_ℓ and B_1, B_2, \dots, B_m there exists a function $R_{\ell, m} : \mathbb{T}^\ell \times \mathbb{T}^m \rightarrow [0, \infty)$ such that

$$E \left[\prod_{j=1}^{\ell} N_{A_j}^{(1)} \prod_{k=1}^m N_{B_k}^{(2)} \right] = \int_{\mathbb{T}^\ell} \int_{\mathbb{T}^m} R_{\ell, m}(\mathbf{x}, \mathbf{z}) d\mu^\ell(\mathbf{x}) d\mu^m(\mathbf{z}),$$

then we call $R_{\ell,m}$ the (ℓ, m) -intensity or correlation function. $R_{\ell,m}$ is dependent on N and X , but we will leave that dependence implicit. Correlation functions are important, for, for instance, computing the probability that a given set contains no particles of a certain species.

Like the RSX model, and by essentially the same proof, the correlation functions can be expressed as the Pfaffian of a $2(\ell + m)$ square antisymmetric matrix whose entries are given in terms of 2×2 matrix kernels which encode information about the interactions between and amongst the different species of particles.

Theorem 3.1. *There exist matrix kernels $\mathbf{K}_N^{1,1}, \mathbf{K}_N^{2,2}, \mathbf{K}_N^{1,2}, \mathbf{K}_N^{2,1} : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{C}^{2 \times 2}$ such that*

$$R_{\ell,m}(\mathbf{x}, \mathbf{z}) = \text{Pf} \begin{bmatrix} \left[\mathbf{K}_N^{1,1}(x_i, x_j) \right]_{i,j=1}^{\ell} & \left[\mathbf{K}_N^{1,2}(x_i, z_n) \right]_{i,n=1}^{\ell,m} \\ \left[\mathbf{K}_N^{2,1}(z_k, x_j) \right]_{k,j=1}^{m,\ell} & \left[\mathbf{K}_N^{2,2}(z_k, z_n) \right]_{k,n=1}^m \end{bmatrix}$$

Using notation which has now become standard, each of these matrix kernels can be written as

$$\mathbf{K}_N^{s,t}(e^{i\theta}, e^{i\psi}) = \begin{bmatrix} DS_N^{s,t}(X; \theta, \psi) & S_N^{s,t}(X; \theta, \psi) \\ -S_N^{t,s}(X; \psi, \theta) & IS_N^{s,t}(X; \theta, \psi) \end{bmatrix}; \quad s, t \in \{1, 2\}, \quad (3.1)$$

where each of the entries is a function $[-\pi, \pi) \times [-\pi, \pi) \rightarrow \mathbb{C}$. Exact formulas for the matrix kernels and their entries will be reported in a subsequent section (Theorem 5.1). To make the X dependence explicit, we will write $\mathbf{K}_N^{s,t}(X; \theta, \psi)$ for the right hand side of (3.1).

Since the total charge of the system is N , we expect that each arc of length $2\pi/N$ will, on average, carry unit charge. Thus, in order to investigate the local behavior between particles we place ourselves on a scale of length $O(N^{-1})$. That is, the local statistics, in a neighborhood of $e^{i\varphi}$ are determined by the scaled kernels

$$\mathbf{K}_N^{s,t} \left(X; \varphi + \frac{2\pi\theta}{N}, \varphi + \frac{2\pi\psi}{N} \right).$$

For purely geometric reasons, the local kernels must be independent of φ , and hence the local statistics are completely determined by $\mathbf{K}_N^{s,t}(X; 2\pi\theta/N, 2\pi\psi/N)$. We wish to investigate the large N limit of these kernels, suitably normalized so that the limit exists. In order to see non-trivial interactions between the two species of particles, we need to scale the fugacity X so that there are $O(N)$ particles of each species. By Theorem 2.2, this goal is met when $X = Nr$ for fixed r , and hence we define the scaling limits of our kernels to be

$$\mathbf{K}^{s,t}(r; \theta, \psi) = \lim_{N \rightarrow \infty} \frac{1}{N} \mathbf{K}_N^{s,t} \left(Nr; \frac{2\pi\theta}{N}, \frac{2\pi\psi}{N} \right).$$

We define the entries of the scaled kernels to be $DS^{s,t}(r, \theta, \psi)$, $S^{s,t}(r, \theta, \psi)$ and $IS^{s,t}(r, \theta, \psi)$, where for instance

$$S^{s,t}(r; \theta, \psi) = \lim_{N \rightarrow \infty} \frac{1}{N} S_N^{s,t} \left(Nr; \frac{2\pi\theta}{N}, \frac{2\pi\psi}{N} \right).$$

Our main result is the evaluation of $\mathbf{K}^{s,t}(r, \theta, \psi)$ and the observation that as $r \rightarrow 0^+$ the resulting Pfaffian point process collapses to that of the Circular Symplectic (and hence

Gaussian Symplectic) ensemble, while as $r \rightarrow \infty$ we recover the kernel for the Circular (and Gaussian) Orthogonal Ensemble. This provides a solvable interpolation between these ensembles, and by universality should provide for the limiting local statistics in the RSX ensemble as well.

Theorem 3.2. *The entries of $\mathbf{K}^{1,1}(r; \theta, \psi)$ are given by*

- $S^{1,1}(r; \theta, \psi) = 4r^2 \int_0^1 \frac{\cos(\pi(\theta - \psi)t)}{4r^2 + t^2} dt$
- $DS^{1,1}(r; \theta, \psi) = ir^2 \int_0^1 \frac{t \sin(\pi(\theta - \psi)t)}{4r^2 + t^2} dt$
- $IS^{1,1}(r; \theta, \psi) = -16ir^2 \int_0^1 \frac{\sin(\pi(\theta - \psi)t)}{4r^2 t + t^3} dt + 2\pi \operatorname{sgn}(\psi - \theta)$

The entries of $\mathbf{K}^{2,2}(r; \theta, \psi)$ are given by

- $S^{2,2}(r; \theta, \psi) = \frac{1}{2} \int_0^1 \frac{t^2 \cos(\pi(\theta - \psi)t)}{4r^2 + t^2} dt$
- $DS^{2,2}(r; \theta, \psi) = \frac{1}{r^2} DS^{1,1}(r; \theta, \psi) = i \int_0^1 \frac{t \sin(\pi(\theta - \psi)t)}{4r^2 + t^2} dt$
- $IS^{2,2}(r; \theta, \psi) = -\frac{i}{4} \int_0^1 \frac{t^3 \sin(\pi(\theta - \psi)t)}{4r^2 + t^2} dt$

The entries of $\mathbf{K}^{1,2}(r; \theta, \psi)$ and $\mathbf{K}^{2,1}(r; \theta, \psi) = -\mathbf{K}^{1,2}(r; \theta, \psi)^\top$ are given by

- $S^{1,2}(r; \theta, \psi) = rS^{2,2}(r; \theta, \psi) = \frac{r}{2} \int_0^1 \frac{t^2 \cos(\pi(\theta - \psi)t)}{4r^2 + t^2} dt$
- $S^{2,1}(r; \theta, \psi) = \frac{1}{r} S^{1,1}(r; \theta, \psi) = 4r \int_0^1 \frac{\cos(\pi(\theta - \psi)t)}{4r^2 + t^2} dt$
- $DS^{1,2}(r; \theta, \psi) = \frac{1}{r} DS^{1,1}(r; \theta, \psi) = ir^2 \int_0^1 \frac{t \sin(\pi(\theta - \psi)t)}{4r^2 + t^2} dt$
- $IS^{1,2}(r; \theta, \psi) = -\frac{2}{r} DS^{1,1}(r; \theta, \psi) = -2ir^2 \int_0^1 \frac{t \sin(\pi(\theta - \psi)t)}{4r^2 + t^2} dt$

As a consistency check, we note that the local spatial density of charge 1 particles is given by

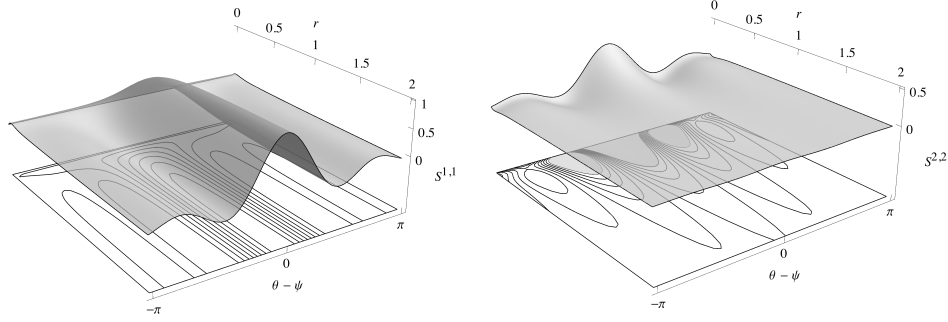
$$S^{1,1}(r; \theta, \theta) = 4r^2 \int_0^1 \frac{1}{4r^2 + t^2} dt = 2r \arctan\left(\frac{1}{2r}\right),$$

which agrees with Theorem 2.2. Similarly, the local spatial density of charge 2 particles is given by

$$S^{2,2}(r; \theta, \theta) = \frac{1}{2} \int_0^1 \frac{t^2}{4r^2 + t^2} dt = \frac{1}{2} - r \arctan\left(\frac{1}{2r}\right).$$

Note that the total local charge density is given by $S^{1,1}(r; \theta, \theta) + 2S^{2,2}(r; \theta, \theta) = 1$ as expected.

The recovery of the kernels for (C/G)OE and (C/G)SE is the content of the following corollary.

Figure 1: $S^{1,1}$ and $S^{2,2}$ as a function of r and $\theta - \psi$.

Corollary 3.3. As $r \rightarrow \infty$,

- $S^{1,1}(r; \theta, \psi) \rightarrow \frac{\sin(\pi(\theta - \psi))}{\pi(\theta - \psi)}$
- $DS^{1,1}(r; \theta, \psi) \rightarrow \frac{i}{4} \left(\frac{\sin(\pi(\theta - \psi))}{\pi^2(\theta - \psi)^2} - \frac{\cos(\pi(\theta - \psi))}{\pi(\theta - \psi)} \right)$
- $IS^{1,1}(r; \theta, \psi) \rightarrow -4i \int_0^1 \frac{\sin(\pi(\theta - \psi)t)}{t} dt + 2\pi \operatorname{sgn}(\psi - \theta)$

and all entries of the remaining kernels go to 0.

As $r \rightarrow 0^+$,

- $S^{2,2}(r; \theta, \psi) \rightarrow \frac{\sin(\pi(\theta - \psi))}{2\pi(\theta - \psi)}$
- $DS^{2,2}(r; \theta, \psi) \rightarrow i \int_0^1 \frac{\sin(\pi(\theta - \psi)t)}{t} dt$
- $IS^{2,2}(r; \theta, \psi) \rightarrow -\frac{i}{4} \left(\frac{\sin(\pi(\theta - \psi))}{\pi^2(\theta - \psi)^2} - \frac{\cos(\pi(\theta - \psi))}{\pi(\theta - \psi)} \right)$
- $IS^{1,1}(r; \theta, \psi) \rightarrow 2\pi \operatorname{sgn}(\psi - \theta)$

and all other entries of the remaining kernels go to 0.

The fact that the $IS^{1,1}$ term ‘stays up’ in the $r \rightarrow 0^+$ limit might seem, on first inspection, suspicious, since in this situation we are (*a posteriori*) tuning the fugacity so that no charge 1 particles appear, and thus we expect the 1, 1 kernel to vanish. However, the local statistics (in the form of the scaled correlations) are determined not by the kernel itself, but the *Pfaffian* of a matrix formed by the kernel. The *Pfaffian* of such a matrix is unchanged

whether or not the limiting $IS^{1,1}$ term stays up or not (this is a rank-1 perturbation, which the Pfaffian cannot detect). In fact, in this limit, any correlation function which has a component measuring interaction from charge 1 particles will be 0, and hence the 1, 1 kernel makes no contribution to the limiting statistics in this instance, independent of the non-zero limit for $IS^{1,1}$. This non-zero entry aside, the kernel entries in both limiting cases can be seen to be essentially equal to the entries of the kernels for (C/G)OE and (C/G)SE, that is equal up to minor changes which do not change the Pfaffians appearing in the limiting correlation functions, appearing in [7].

4 Acknowledgments

After posting this manuscript to the arXiv, I (Sinclair) received the following email from Peter Forrester:

Dear Chris,

I'm sure that back in 2010 you were aware that I introduced the model of your recent arXiv posting in two papers published in 1984, and that in my book I extended this work to the computation of the general correlation functions. Now that your memory has been refreshed, I hope that you'll appropriately modify your posting.

We thank Peter for kindly pointing out his previous work in the area and are happy to acknowledge the existence of [6], [3] and [4, §6.7]. It should be remarked that, while the model considered here is closely related to Forrester's model, in our model, the number of each type of particle is a random variable. It is unsurprising that, in the large N limit with fugacity tuned so that there is a positive proportion of each species, the scaled correlation functions match Forrester's in the limit where L and M (which are non-random in his model) attain some fixed ratio. Of course, the introduction of the fugacity introduces new and interesting questions (some of which we resolve here) which do not arise in the model with a fixed number of each species.

Finally, a history of this manuscript is in order. This is one chapter of the first author's (Shum's) Ph.D. thesis [10], in which several solvable interpolations between classical random matrix ensembles are studied. He proposed and solved, quite independently, the problems in this manuscript after reading [9], which gives the analogous ensemble on the line, and left open the determination of the scaling limits of the kernels in that case. Of course, universality suggests the scaled (bulk) kernels in that case should agree with those presented here (and apparently previously in [4]) which was the primary motivation for the present work.

My role, besides giving advice and encouragement, has been to prepare the manuscript for publication since Chris Shum has since left academia.

5 Proofs

We will restrict ourselves to the case where N is even. The odd N cases are more tedious, and we refer to the first author's Ph.D. thesis for details [10] (see also [12] [13] [5] [2] [1] where methods of deriving quantities for odd N from those for even N are discussed for various ensembles). Note that, when N is even, so too is $L = N - 2M$.

5.1 The Proof of Theorem 3.2

The proof of this Theorem follows from standard methods in random matrix theory. We will give only the basic details and point the reader to [9, §4.5] (see also [12] on which the ideas in [9] rely).

For each non-negative pair of integers L, M with $L + 2M = N$, let $\mathbf{V}_{L,M}(\xi, \zeta)$ be the confluent Vandermonde matrix

$$\mathbf{V}_{L,M}(\xi, \zeta) = \begin{bmatrix} 1 & \cdots & 1 & 1 & 0 & \cdots & 1 & 0 \\ \xi_1 & & \xi_L & \zeta_1 & 1 & & \zeta_M & 1 \\ \xi_1^2 & \cdots & \xi_L^2 & \zeta_1^2 & 2\zeta_1 & \cdots & \zeta_M^2 & 2\zeta_M \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \xi_1^{n-1} & \cdots & \xi_L^{n-1} & \zeta_1^{n-1} & (n-1)\zeta_1^{n-2} & \cdots & \zeta_M^{n-1} & (n-1)\zeta_M^{n-2} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \xi_1^{N-1} & \cdots & \xi_L^{N-1} & \zeta_1^{N-1} & (N-1)\zeta_1^{N-2} & \cdots & \zeta_M^{N-1} & (N-1)\zeta_M^{N-2} \end{bmatrix}.$$

It is well-known, [8], that

$$|\det \mathbf{V}_{L,M}(\xi, \zeta)| = \prod_{k < \ell} |\xi_\ell - \xi_k| \prod_{m < n} |\zeta_n - \zeta_m|^4 \prod_{\ell=1}^L \prod_{m=1}^M |\zeta_m - \xi_\ell|^2 = e^{-E_{L,M}(\xi, \zeta)}.$$

Moreover, for $\zeta, \xi \in \mathbb{T}$, define $\text{sgn}(\zeta - \xi) := \text{sgn}(\text{Arg}(\zeta) - \text{Arg}(\xi))$ for the branch of the argument taking values in $[-\pi, \pi)$. Then,

$$|\zeta - \xi| = -i(\zeta - \xi)\xi^{-1/2}\zeta^{-1/2}\text{sgn}(\zeta - \xi)$$

and hence,

$$e^{-E_{L,M}(\xi, \zeta)} = \left\{ \prod_{\ell=1}^L e^{\frac{\pi i}{4}} \xi_\ell^{\frac{1-N}{2}} \prod_{m=1}^M \zeta_m^{2-N} \prod_{k < \ell} \text{sgn}(\xi_\ell - \xi_k) \right\} \det \mathbf{V}_{L,M}(\xi, \zeta)$$

Setting

$$d\mu_1(\xi) = e^{\frac{\pi i}{4}} \xi^{-\frac{N-1}{2}} d\mu(\xi) \quad \text{and} \quad d\mu_2(\zeta) = \zeta^{-N-1} d\mu(\zeta),$$

we find

$$Z_N(X) = \sum_{(L,M)} \frac{X^L}{L!M!} \int_{\mathbb{T}^L} \int_{\mathbb{T}^M} \left\{ \prod_{k < \ell} \text{sgn}(\xi_\ell - \xi_k) \right\} \det \mathbf{V}_{L,M}(\xi, \zeta) d\mu_1^L(\xi) d\mu_2^M(\zeta).$$

Here we direct the reader to [9, §4.5], where by replacing all instances of \mathbb{R} with \mathbb{T} , we obtain

$$Z_N(X) = \text{Pf} \left(X^2 \mathbf{A}_N + \mathbf{B}_N \right), \quad (5.1)$$

where

$$\mathbf{A}_N = \left[\int_{\mathbb{T}} \int_{\mathbb{T}} \xi_1^m \xi_2^n \text{sgn}(\xi_2 - \xi_1) d\mu_1(\xi_1) d\mu_2(\xi_2) \right]_{m,n=1}^N$$

and

$$\mathbf{B}_N = \left[(m-n) \int_{\mathbb{T}} \zeta^m \zeta^n d\mu_2(\zeta) \right].$$

It remains to evaluate the integrals appearing in \mathbf{A}_N and \mathbf{B}_N and to evaluate the Pfaffian appearing in (5.1).

Clearly,

$$(m-n) \int_{\mathbb{T}} \zeta^{m+n-N-1} d\mu(\zeta) = \begin{cases} 2\pi(N-2n+1) & \text{if } n+m = N+1; \\ 0 & \text{otherwise.} \end{cases}$$

An (only slightly) more involved calculation shows

$$-i \int_{\mathbb{T}} \int_{\mathbb{T}} \xi_1^{\frac{2m-N-1}{2}} \xi_2^{\frac{2n-N-1}{2}} \operatorname{sgn}(\xi_2 - \xi_1) d\mu(\xi_1) d\mu(\xi_2) = \begin{cases} \frac{8\pi}{(N-2n+1)} & \text{if } n+m = N+1; \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $X^2 \mathbf{A}_N + \mathbf{B}_N$ is an antisymmetric matrix with non-zero entries only on the antidiagonal, and hence

$$\begin{aligned} \operatorname{Pf}(X^2 \mathbf{A}_N + \mathbf{B}_N) &= \prod_{n=1}^{N/2} \left(X^2 \frac{8\pi}{N-2n+1} + 2\pi(N-2n+1) \right) \\ &= (2\pi)^{N/2} \prod_{n=1}^{N/2} \frac{(2X)^2 + (2n-1)^2}{2n-1}. \end{aligned}$$

To prove Corollary 2.1, note that

$$\lim_{N \rightarrow \infty} \frac{Z_N(TX)}{Z_N(X)} = \lim_{N \rightarrow \infty} \prod_{n=1}^{N/2} \left(1 + \frac{(2TX)^2}{(2n-1)^2} \right) \bigg/ \prod_{n=1}^{N/2} \left(1 + \frac{(2X)^2}{(2n-1)^2} \right),$$

this together with the infinite product formula of cosine,

$$\cos(x) = \prod_{n=1}^{\infty} \left(1 - \frac{4x^2}{\pi^2(2n-1)^2} \right),$$

produces the result.

5.2 The Proof of Theorem 2.2

The characteristic function and cumulant generating functions of $L_N(X)$ are given by

$$\varphi_N(X; t) = \frac{Z_N(Xe^{it})}{Z_N(X)} = \prod_{n=1}^{N/2} \frac{(2Xe^{it})^2 + (2n-1)^2}{(2X)^2 + (2n-1)^2},$$

and

$$\begin{aligned} K_N(X; t) &= \log \varphi_N(X; t) \\ &= \sum_{n=1}^{N/2} \log \left[(2Xe^{it})^2 + (2n-1)^2 \right] - \sum_{n=1}^{N/2} \log \left[(2X)^2 + (2n-1)^2 \right]. \end{aligned}$$

It follows that

$$E[L_N(X)] = \frac{1}{i} \frac{d}{dt} K_N(X, t) \Big|_{t=0} = \sum_{n=1}^{N/2} \frac{8X^2}{4X^2 + (2n-1)^2}.$$

Thus,

$$\frac{1}{N} E[L_N(Nr)] = \frac{2}{N} \sum_{n=1}^{N/2} \frac{(2r)^2}{(2r)^2 + \left(\frac{2n-1}{N}\right)^2}.$$

Note that the latter sum is a Riemann sum, and hence

$$\lim_{N \rightarrow \infty} \frac{1}{N} E[L_N(Nr)] = \int_0^1 \frac{(2r)^2}{(2r)^2 + t^2} dt = 2r \arctan\left(\frac{1}{2r}\right).$$

Similarly,

$$\text{var}(L_N(X)) = -\frac{d^2}{dt^2} K_N(X, t) \Big|_{t=0} = \sum_{n=1}^{N/2} \left(\frac{4X(2n-1)}{4X^2 + (2n-1)^2} \right)^2.$$

And,

$$\frac{1}{N} \text{var}(L_N(Nr)) = \frac{4}{N} \sum_{n=1}^{N/2} \frac{(2r)^2 \left(\frac{2n-1}{N}\right)^2}{\left((2r)^2 + \left(\frac{2n-1}{N}\right)^2\right)^2}$$

which again is a Riemann sum, and hence

$$\lim_{N \rightarrow \infty} \frac{1}{N} \text{var}(L_N(Nr)) = 2 \int_0^1 \frac{(2rt)^2}{((2r)^2 + t^2)^2} dt = 2r \arctan\left(\frac{1}{2r}\right) - \frac{4r^2}{1 + 4r^2}.$$

5.3 The Proof of Theorem 2.3

We take the logarithm of the characteristic function

$$E \left[\exp \left(it \frac{L_N(Nr) - \mu_N}{\sigma_N} \right) \right],$$

but first we must define a branch of the logarithm. Because $\mu_N/\sigma_N = O(N^{-\frac{1}{2}})$, for fixed $t \in \mathbb{R}$, there exists N large enough such that $-\pi < t\mu_N/\sigma_N < \pi$, and so take the branch to

be the negative real line. Then for this N ,

$$\begin{aligned}
& \log \left[E \left[\exp \left(it \frac{L_N(Nr) - \mu_N}{\sigma_N} \right) \right] \right] \\
&= \log \left[E \left[\exp \left(it \frac{L_N(Nr)}{\sigma_N} \right) \right] \right] + \log \left[E \left[\exp \left(-it \frac{\mu_N}{\sigma_N} \right) \right] \right] \\
&= \log \left[\prod_{n=1}^{\frac{N}{2}} \frac{\left(2Nr \exp \left(\frac{it}{\sigma_N} \right) \right)^2 + (2n-1)^2}{(2Nr)^2 + (2n-1)^2} \right] - it \frac{\mu_N}{\sigma_N} \\
&= \sum_{n=1}^{\frac{N}{2}} \log \left[\frac{\left(2Nr \exp \left(\frac{it}{\sigma_N} \right) \right)^2 + (2n-1)^2}{(2Nr)^2 + (2n-1)^2} \right] - it \frac{\mu_N}{\sigma_N} \\
&= \sum_{n=1}^{\frac{N}{2}} \log \left[\frac{\exp \left(\frac{2it}{\sigma_N} \right) + \left(\frac{2n-1}{2Nr} \right)^2}{1 + \left(\frac{2n-1}{2Nr} \right)^2} \right] - it \frac{\mu_N}{\sigma_N}. \tag{5.2}
\end{aligned}$$

We claim that the above expression (5.2) may be replaced by

$$\sum_{n=1}^{\frac{N}{2}} \left[\left(\frac{\frac{2it}{\sigma_N} + \frac{1}{2!} \left(\frac{2it}{\sigma_N} \right)^2}{1 + \left(\frac{2n-1}{2Nr} \right)^2} \right) - \frac{1}{2} \left(\frac{\frac{2it}{\sigma_N} + \frac{1}{2!} \left(\frac{2it}{\sigma_N} \right)^2}{1 + \left(\frac{2n-1}{2Nr} \right)^2} \right)^2 \right] - it \frac{\mu_N}{\sigma_N}, \tag{5.3}$$

i.e. (5.2) and (5.3) converge to the same limit as $N \rightarrow \infty$. To see this, first take the Taylor expansion in (5.2) for the logarithm, which is valid since for large enough N , the value inside the logarithm is less than 2. So (5.2) is equal to

$$\begin{aligned}
& \sum_{n=1}^{\frac{N}{2}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left(\frac{\exp \left(\frac{2it}{\sigma_N} \right) + \left(\frac{2n-1}{2Nr} \right)^2}{1 + \left(\frac{2n-1}{2Nr} \right)^2} - 1 \right)^k - it \frac{\mu_N}{\sigma_N} \\
&= \sum_{n=1}^{\frac{N}{2}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left(\frac{\exp \left(\frac{2it}{\sigma_N} \right) - 1}{1 + \left(\frac{2n-1}{2Nr} \right)^2} \right)^k - it \frac{\mu_N}{\sigma_N}. \tag{5.4}
\end{aligned}$$

So the absolute value of the difference between (5.4) and (5.3) is

$$\begin{aligned}
& \left| \sum_{n=1}^{\frac{N}{2}} \left[\left(\frac{\frac{2it}{\sigma_N} + \frac{1}{2!} \left(\frac{2it}{\sigma_N} \right)^2}{1 + \left(\frac{2n-1}{2Nr} \right)^2} \right) - \left(\frac{\exp \left(\frac{2it}{\sigma_N} \right) - 1}{1 + \left(\frac{2n-1}{2Nr} \right)^2} \right) \right] \right. \\
& \quad \left. - \sum_{n=1}^{\frac{N}{2}} \left[\frac{1}{2} \left(\frac{\frac{2it}{\sigma_N} + \frac{1}{2!} \left(\frac{2it}{\sigma_N} \right)^2}{1 + \left(\frac{2n-1}{2Nr} \right)^2} \right)^2 - \frac{1}{2} \left(\frac{\exp \left(\frac{2it}{\sigma_N} \right) - 1}{1 + \left(\frac{2n-1}{2Nr} \right)^2} \right)^2 \right] \right. \\
& \quad \left. + \sum_{n=1}^{\frac{N}{2}} \sum_{k=3}^{\infty} \frac{(-1)^{k+1}}{k} \left(\frac{\exp \left(\frac{2it}{\sigma_N} \right) - 1}{1 + \left(\frac{2n-1}{2Nr} \right)^2} \right)^k \right|. \quad (5.5)
\end{aligned}$$

We will show that the absolute values of the three terms converge to 0 as $N \rightarrow \infty$.

The third term of (5.5) is

$$\begin{aligned}
& \left| \sum_{n=1}^{\frac{N}{2}} \sum_{k=3}^{\infty} \frac{(-1)^{k+1}}{k} \left(\frac{\exp \left(\frac{2it}{\sigma_N} \right) - 1}{1 + \left(\frac{2n-1}{2Nr} \right)^2} \right)^k \right| \\
& \leq \sum_{n=1}^{\frac{N}{2}} \left(\left| \frac{\exp \left(\frac{2it}{\sigma_N} \right) - 1}{1 + \left(\frac{2n-1}{2Nr} \right)^2} \right|^3 \left| \sum_{k=1}^{\infty} \frac{(-1)^{k+4}}{k+3} \left(\frac{\exp \left(\frac{2it}{\sigma_N} \right) - 1}{1 + \left(\frac{2n-1}{2Nr} \right)^2} \right)^k \right| \right).
\end{aligned}$$

Using the Taylor expansion for the exponential function, there exists $A > 0$ such that

$$\left| \frac{\exp \left(\frac{2it}{\sigma_N} \right) - 1}{1 + \left(\frac{2n-1}{2Nr} \right)^2} \right|^3 \leq \frac{A |t|^3}{\sigma_N^3 \left(1 + \left(\frac{2n-1}{2Nr} \right)^2 \right)^3} \leq \frac{A |t|^3}{\sigma_N^3} = \frac{A |t|^3}{\sigma^3 N^3}. \quad (5.6)$$

Next,

$$\begin{aligned}
& \left| \sum_{k=1}^{\infty} \frac{(-1)^{k+4}}{k+3} \left(\frac{\exp \left(\frac{2it}{\sigma_N} \right) - 1}{1 + \left(\frac{2n-1}{2Nr} \right)^2} \right)^k \right| \leq \left| \log \left[\frac{\exp \left(\frac{2it}{\sigma_N} \right) + \left(\frac{2n-1}{2Nr} \right)^2}{1 + \left(\frac{2n-1}{2Nr} \right)^2} \right] \right| \\
& \leq \left| \log \left[\exp \left(\frac{2it}{\sigma_N} \right) + \left(\frac{2N-1}{2Nr} \right)^2 \right] \right|. \quad (5.7)
\end{aligned}$$

Thus (5.6) and (5.7) show that

$$\begin{aligned}
& \sum_{n=1}^{\frac{N}{2}} \left(\left| \frac{\exp\left(\frac{2it}{\sigma_N}\right) - 1}{1 + \left(\frac{2n-1}{2Nr}\right)^2} \right|^3 \left| \sum_{k=1}^{\infty} \frac{(-1)^{k+4}}{k+3} \left(\frac{\exp\left(\frac{2it}{\sigma_N}\right) - 1}{1 + \left(\frac{2n-1}{2Nr}\right)^2} \right)^k \right| \right) \\
& \leq \sum_{n=1}^{\frac{N}{2}} \frac{A|t|^3}{\sigma^3 N^3} \cdot \left| \log \left[\exp\left(\frac{2it}{\sigma_N}\right) + \left(\frac{2N-1}{2Nr}\right)^2 \right] \right| \\
& = \frac{A|t|^3}{2\sigma^3 N^2} \cdot \left| \log \left[\exp\left(\frac{2it}{\sigma_N}\right) + \left(\frac{2N-1}{2Nr}\right)^2 \right] \right|,
\end{aligned}$$

which converges to 0 as $N \rightarrow \infty$.

Using the same argument from (5.6), we have the estimate on the first term of (5.5)

$$\left| \sum_{n=1}^{\frac{N}{2}} \left[\left(\frac{\frac{2it}{\sigma_N} + \frac{1}{2!} \left(\frac{2it}{\sigma_N} \right)^2}{1 + \left(\frac{2n-1}{2Nr} \right)^2} \right) - \left(\frac{\exp\left(\frac{2it}{\sigma_N}\right) - 1}{1 + \left(\frac{2n-1}{2Nr} \right)^2} \right) \right] \right| \leq \sum_{n=1}^{\frac{N}{2}} \frac{A|t|^3}{\sigma^3 N^3} = \frac{A|t|^3}{2\sigma^3 N^2},$$

which also goes to 0 as $N \rightarrow \infty$.

Finally, the summands of the second term in (5.5) can be combined as

$$\frac{1}{2} \left[\frac{\left(\frac{2it}{\sigma_N} + \frac{1}{2!} \left(\frac{2it}{\sigma_N} \right)^2 - \exp\left(\frac{2it}{\sigma_N}\right) + 1 \right)^2 + 2 \left(\frac{2it}{\sigma_N} + \frac{1}{2!} \left(\frac{2it}{\sigma_N} \right)^2 \right) \left(\exp\left(\frac{2it}{\sigma_N}\right) - 1 \right)}{\left(1 + \left(\frac{2n-1}{2Nr} \right)^2 \right)^2} \right].$$

Again, using the same argument as in (5.6), there exists $A > 0$ such that the absolute value of the first part of the sum is

$$\left| \frac{\left(\frac{2it}{\sigma_N} + \frac{1}{2!} \left(\frac{2it}{\sigma_N} \right)^2 - \exp\left(\frac{2it}{\sigma_N}\right) + 1 \right)^2}{\left(1 + \left(\frac{2n-1}{2Nr} \right)^2 \right)^2} \right| \leq \frac{A^2 t^6}{\sigma^6 N^6 \left(1 + \left(\frac{2n-1}{2Nr} \right)^2 \right)^2} \leq \frac{A^2 t^6}{\sigma^6 N^6}, \quad (5.8)$$

and there exists $B > 0$ such that the absolute value of the second part of the sum is

$$\begin{aligned}
\left| \frac{2 \left(\frac{2it}{\sigma_N} + \frac{1}{2!} \left(\frac{2it}{\sigma_N} \right)^2 \right) \left(\exp \left(\frac{2it}{\sigma_N} \right) - 1 \right)}{\left(1 + \left(\frac{2n-1}{2Nr} \right)^2 \right)^2} \right| &\leq \left| \frac{2 \left(\frac{2it}{\sigma_N} + \frac{1}{2!} \left(\frac{2it}{\sigma_N} \right)^2 \right) \left(\frac{2Bit}{\sigma_N} \right)}{\left(1 + \left(\frac{2n-1}{2Nr} \right)^2 \right)^2} \right| \\
&= \left| \frac{4Bt^2 \left(1 + \frac{1}{2!} \left(\frac{2it}{\sigma_N} \right) \right)}{\sigma^2 N^2 \left(1 + \left(\frac{2n-1}{2Nr} \right)^2 \right)^2} \right| \\
&\leq \left| \frac{4Bt^2 \left(1 + \frac{it}{\sigma_N} \right)}{\sigma^2 N^2} \right|. \tag{5.9}
\end{aligned}$$

Thus, using (5.8) and (5.9) the absolute value of the second summand

$$\begin{aligned}
&\left| \sum_{n=1}^{\frac{N}{2}} \left[\frac{1}{2} \left(\frac{\frac{2it}{\sigma_N} + \frac{1}{2!} \left(\frac{2it}{\sigma_N} \right)^2}{1 + \left(\frac{2n-1}{2Nr} \right)^2} \right)^2 - \frac{1}{2} \left(\frac{\exp \left(\frac{2it}{\sigma_N} \right) - 1}{1 + \left(\frac{2n-1}{2Nr} \right)^2} \right)^2 \right] \right| \\
&\leq \frac{1}{2} \sum_{n=1}^{\frac{N}{2}} \left(\frac{A^2 t^6}{\sigma^6 N^6} + \left| \frac{4Bt^2 \left(1 + \frac{it}{\sigma_N} \right)}{\sigma^2 N^2} \right| \right) \\
&= \frac{1}{2\sigma^2 N^2} \sum_{n=1}^{\frac{N}{2}} \left(\frac{A^2 t^6}{\sigma^4 N^4} + \left| 4Bt^2 \left(1 + \frac{it}{\sigma_N} \right) \right| \right) \\
&= \frac{1}{4\sigma^2 N} \left(\frac{A^2 t^6}{\sigma^4 N^4} + \left| 4Bt^2 \left(1 + \frac{it}{\sigma_N} \right) \right| \right)
\end{aligned}$$

which converges to 0 as $N \rightarrow \infty$. Thus, (5.5) converges to 0, so (5.2) and (5.3) converge to the same limit.

We rewrite (5.3) as

$$\begin{aligned}
& \sum_{n=1}^{\frac{N}{2}} \left[\left(\frac{\frac{2it}{\sigma_N} + \frac{1}{2!} \left(\frac{2it}{\sigma_N} \right)^2}{1 + \left(\frac{2n-1}{2Nr} \right)^2} \right) - \frac{1}{2} \left(\frac{\frac{2it}{\sigma_N} + \frac{1}{2!} \left(\frac{2it}{\sigma_N} \right)^2}{1 + \left(\frac{2n-1}{2Nr} \right)^2} \right)^2 \right] - it \frac{\mu_N}{\sigma_N} \\
&= \underbrace{\sum_{n=1}^{\frac{N}{2}} \left[-\frac{2t^2}{\sigma_N^2 \left(1 + \left(\frac{2n-1}{2Nr} \right)^2 \right)} + \frac{2t^2}{\sigma_N^2 \left(1 + \left(\frac{2n-1}{2Nr} \right)^2 \right)^2} \right]}_{(*)} \\
&+ \underbrace{\sum_{n=1}^{\frac{N}{2}} \left[\frac{2it}{\sigma_N \left(1 + \left(\frac{2n-1}{2Nr} \right)^2 \right)} + \frac{4it^3}{\sigma_N^3 \left(1 + \left(\frac{2n-1}{2Nr} \right)^2 \right)^2} - \frac{2t^4}{\sigma_N^4 \left(1 + \left(\frac{2n-1}{2Nr} \right)^2 \right)^2} \right]}_{(**)} - it \frac{\mu_N}{\sigma_N}.
\end{aligned}$$

We will show that the expressions (*) and (**) converge to $-\frac{t^2}{2}$ and 0, respectively. First we define

$$\mu = \frac{\mu_N}{N} = 2r \arctan \left(\frac{1}{2r} \right),$$

and let σ be the positive number such that

$$\sigma^2 = \frac{\sigma_N^2}{N} = \left(2r \arctan \left(\frac{1}{2r} \right) - \frac{4r^2}{1 + 4r^2} \right).$$

Then,

$$\begin{aligned}
-\frac{2t^2}{\sigma_N^2} \sum_{n=1}^{\frac{N}{2}} \frac{1}{1 + \left(\frac{2n-1}{2Nr} \right)^2} &= -\frac{t^2}{\sigma^2} \cdot \frac{2}{N} \sum_{n=1}^{\frac{N}{2}} \frac{1}{1 + \left(\frac{2n-1}{2Nr} \right)^2} \\
&\rightarrow -\frac{t^2}{\sigma^2} \int_0^1 \frac{1}{1 + \left(\frac{t}{2r} \right)^2} dt = -2r \arctan \left(\frac{1}{2r} \right) \cdot \frac{t^2}{\sigma^2},
\end{aligned}$$

and

$$\begin{aligned}
\frac{2t^2}{\sigma_N^2} \sum_{n=1}^{\frac{N}{2}} \frac{1}{\left(1 + \left(\frac{2n-1}{2Nr} \right)^2 \right)^2} &= \frac{t^2}{\sigma^2} \cdot \frac{2}{N} \sum_{n=1}^{\frac{N}{2}} \frac{1}{\left(1 + \left(\frac{2n-1}{2Nr} \right)^2 \right)^2} \\
&\rightarrow \frac{t^2}{\sigma^2} \int_0^1 \frac{1}{\left(1 + \left(\frac{t}{2r} \right)^2 \right)^2} dt \\
&= \left(2r \arctan \left(\frac{1}{2r} \right) + \frac{4r^2}{1 + 4r^2} \right) \frac{t^2}{2\sigma^2}.
\end{aligned}$$

Thus, changing $N/2 \mapsto N$,

$$\begin{aligned} \sum_{n=1}^N \left[-\frac{2t^2}{\sigma_{2N}^2 \left(1 + \left(\frac{2n-1}{4Nr}\right)^2\right)} + \frac{2t^2}{\sigma_{2N}^2 \left(1 + \left(\frac{2n-1}{4Nr}\right)^2\right)^2} \right] \\ \longrightarrow \left(2r \arctan\left(\frac{1}{2r}\right) + \frac{4r^2}{1+4r^2} \right) \frac{t^2}{2\sigma^2} - 2r \arctan\left(\frac{1}{2r}\right) \cdot \frac{t^2}{\sigma^2} \\ = -\frac{t^2}{2}. \end{aligned}$$

Next,

$$\begin{aligned} \frac{2it}{\sigma_N} \sum_{n=1}^{\frac{N}{2}} \frac{1}{1 + \left(\frac{2n-1}{2Nr}\right)^2} - it \frac{\mu_N}{\sigma_N} &= \frac{\sqrt{N}it}{\sigma} \left(\frac{2}{N} \sum_{n=1}^{\frac{N}{2}} \frac{1}{1 + \left(\frac{2n-1}{2Nr}\right)^2} - \mu \right) \\ &= \frac{\sqrt{N}it}{\sigma} \left(\frac{2}{N} \sum_{n=1}^{\frac{N}{2}} \frac{1}{1 + \left(\frac{2n-1}{2Nr}\right)^2} - \int_0^1 \frac{1}{1 + \left(\frac{t}{2r}\right)^2} dt \right). \end{aligned}$$

It is a well-known calculus fact that the error term of the Riemann sum using the midpoint rule is $O(N^{-2})$ if the integrand is twice differentiable. That is,

$$\frac{2it}{\sigma_N} \sum_{n=1}^{\frac{N}{2}} \left[\frac{1}{1 + \left(\frac{2n-1}{2Nr}\right)^2} \right] - it \frac{\mu_N}{\sigma_N} \longrightarrow 0.$$

Similarly,

$$-\frac{2t^4}{\sigma_N^4} \sum_{n=1}^{\frac{N}{2}} \frac{1}{\left(1 + \left(\frac{2n-1}{2Nr}\right)^2\right)^2} = -\frac{t^4}{N\sigma^4} \cdot \frac{2}{N} \sum_{n=1}^{\frac{N}{2}} \frac{1}{\left(1 + \left(\frac{2n-1}{2Nr}\right)^2\right)^2} \longrightarrow 0$$

and since the sums converge to finite integrals,

$$\frac{4it^3}{\sigma_N^3} \sum_{n=1}^{\frac{N}{2}} \frac{1}{\left(1 + \left(\frac{2n-1}{2Nr}\right)^2\right)^2} = \frac{2it^3}{\sqrt{N}\sigma^3} \cdot \frac{2}{N} \sum_{n=1}^{\frac{N}{2}} \frac{1}{\left(1 + \left(\frac{2n-1}{2Nr}\right)^2\right)^2} \longrightarrow 0$$

This shows that

$$\log \left[\mathbb{E} \left[\exp \left(it \frac{L_N(Nr) - \mu_N}{\sigma_N} \right) \right] \right] \longrightarrow -\frac{t^2}{2},$$

establishing the Central Limit Theorem for $L_N(Nr)$.

5.4 The Proof of Theorem 3.1

Theorem 3.1 follows *mutadis mutandis* from the proof of Theorem 3.3 in [9] (which is based on [2, §7], [13] and ultimately [14]). In order to appeal directly to [9, §4.6], we define for indeterminants $a_1, \dots, a_N, b_1, \dots, b_N$ and $x_1, \dots, x_N, z_1, \dots, z_N \in \mathbb{T}$ the measures

$$\eta_1(\xi) = \sum_{n=1}^N a_n \delta(\xi - x_n) \quad \text{and} \quad \eta_2(\zeta) = \sum_{n=1}^N b_n \delta(\zeta - z_n),$$

where $\delta(0)$ is probability measure with unit mass at 0. From this we define the measures

$$d\nu_1(\xi) = e^{\frac{\pi i}{4}} \xi^{-\frac{N-1}{2}} (d\mu(\xi) + d\eta_1(\xi)) \quad \text{and} \quad d\nu_2(\zeta) = \zeta^{-N-1} (d\mu(\zeta) + d\eta_2(\zeta)).$$

Then, $R_{\ell,m}(x_1, \dots, x_\ell, z_1, \dots, z_m)$ is the coefficient of $a_1 \cdots a_\ell b_1 \cdots b_m$ in

$$Z_N^{\nu_1, \nu_2}(X) := \sum_{L+2M=N} \frac{1}{L!M!} |\det \mathbf{V}_{L,M}(\boldsymbol{\xi}, \boldsymbol{\zeta})| d\nu_1^L(\boldsymbol{\xi}) d\nu_2^M(\boldsymbol{\zeta}). \quad (5.10)$$

Computing the right-hand side of (5.10) as a Pfaffian and reading off the desired coefficient allows us to express $R_{\ell,m}(\mathbf{x}, \mathbf{z})$ as the Pfaffian of a matrix of the form

$$\text{Pf} \begin{bmatrix} \left[\mathbf{K}_N^{1,1}(x_i, x_j) \right]_{i,j=1}^{\ell} & \left[\mathbf{K}_N^{1,2}(x_i, z_n) \right]_{i,n=1}^{\ell, m} \\ \left[\mathbf{K}_N^{2,1}(z_k, x_j) \right]_{k,j=1}^{m, \ell} & \left[\mathbf{K}_N^{2,2}(z_k, z_n) \right]_{k,n=1}^m \end{bmatrix}$$

In fact, the calculation in [9] yields the explicit form of $\mathbf{K}_N^{1,1}$, $\mathbf{K}_N^{2,2}$ and $\mathbf{K}_N^{1,2}$ and $\mathbf{K}_N^{2,1} = -(\mathbf{K}_N^{1,2})^\top$. We report the entries of these matrix kernels here, but leave the details of the calculation to the reader.

Theorem 5.1. *Assuming N is even, and with notation as in (3.1), the entries of $\mathbf{K}_N^{1,1}(X; \theta, \psi)$ are given by*

- $S_N^{1,1}(X; \theta, \psi) = \frac{4X^2}{\pi} \sum_{n=1}^{N/2} \frac{\cos\left(\left(n - \frac{1}{2}\right)(\theta - \psi)\right)}{(2X)^2 + (2n-1)^2}$
- $DS_N^{1,1}(X; \theta, \psi) = \frac{iX^2}{\pi} \sum_{n=1}^{N/2} \frac{(2n-1) \sin\left(\left(n - \frac{1}{2}\right)(\theta - \psi)\right)}{(2X)^2 + (2n-1)^2}$
- $IS_N^{1,1}(X; \theta, \psi) = -\frac{16iX^2}{\pi} \sum_{n=1}^{N/2} \frac{\sin\left(\left(n - \frac{1}{2}\right)(\theta - \psi)\right)}{(2n-1)((2X)^2 + (2n-1)^2)} + \text{sgn}(\psi - \theta)$

The entries of $\mathbf{K}_N^{2,2}(X; \theta, \psi)$ are given by

- $S_N^{2,2}(X; \theta, \psi) = \frac{1}{2\pi} \sum_{n=1}^{N/2} \frac{(2n-1)^2 \cos\left(\left(n - \frac{1}{2}\right)(\theta - \psi)\right)}{(2X)^2 + (2n-1)^2}$

- $DS_N^{2,2}(X; \theta, \psi) = \frac{i}{\pi} \sum_{n=1}^{N/2} \frac{(2n-1) \sin\left(\left(n - \frac{1}{2}\right)(\theta - \psi)\right)}{(2X)^2 + (2n-1)^2}$
- $IS_N^{2,2}(X; \theta, \psi) = \frac{-i}{4\pi} \sum_{n=1}^{N/2} \frac{(2n-1)^3 \sin\left(\left(n - \frac{1}{2}\right)(\theta - \psi)\right)}{(2X)^2 + (2n-1)^2}$

The entries of $\mathbf{K}_N^{1,2}(X; \theta, \psi)$ and $\mathbf{K}_N^{2,1}(X; \theta, \psi) = -\mathbf{K}_N^{1,2}(X; \psi, \theta)^\top$ are given by

- $S_N^{1,2}(X; \theta, \psi) = \frac{X}{2\pi} \sum_{n=1}^{N/2} \frac{(2n-1)^2 \cos\left(\left(n - \frac{1}{2}\right)(\theta - \psi)\right)}{(2X)^2 + (2n-1)^2}$
- $S_N^{2,1}(X; \theta, \psi) = \frac{4X}{\pi} \sum_{n=1}^{N/2} \frac{\cos\left((2n-1)\left(n - \frac{1}{2}\right)(\theta - \psi)\right)}{(2X)^2 + (2n-1)^2}$
- $DS_N^{1,2}(X; \theta, \psi) = \frac{iX}{\pi} \sum_{n=1}^{N/2} \frac{(2n-1) \sin\left(\left(n - \frac{1}{2}\right)(\theta - \psi)\right)}{(2X)^2 + (2n-1)^2}$
- $IS_N^{1,2}(X; \theta, \psi) = -\frac{2iX}{\pi} \sum_{n=1}^{N/2} \frac{(2n-1) \sin\left(\left(n - \frac{1}{2}\right)(\theta - \psi)\right)}{(2X)^2 + (2n-1)^2}$

5.5 The Proof of Theorem 3.2

Write $\mathbf{K}_N^{(\ell,m)}(X; \theta, \psi)$ for the $2\ell + 2m$ square matrix

$$\mathbf{K}_N^{(\ell,m)}(X; \theta, \psi) \begin{bmatrix} \left[\mathbf{K}_N^{1,1}(\theta_i, \theta_j) \right]_{i,j=1}^{\ell} & \left[\mathbf{K}_N^{1,2}(\theta_i, \psi_n) \right]_{i,n=1}^{\ell,m} \\ \left[\mathbf{K}_N^{2,1}(\psi_k, \theta_j) \right]_{k,j=1}^{m,\ell} & \left[\mathbf{K}_N^{2,2}(\psi_k, \psi_n) \right]_{k,n=1}^m \end{bmatrix}$$

The Pfaffian of this matrix, of course, yields the ℓ, m correlation function (in terms of the arguments of the variables).

In order to calculate the large N limit of the correlations when the fugacity is tuned so that there are a positive proportion of both species in the limit, we set $X = Nr$ for $r > 0$. Using the well-known Pfaffian identity,

$$\text{Pf}(\mathbf{B}\mathbf{A}\mathbf{B}^\top) = \det \mathbf{B} \text{Pf} \mathbf{A}$$

we may multiply $\mathbf{K}_N^{(\ell,m)}(X; \theta, \psi)$ on the left and right by the diagonal matrix

$$\mathbf{D} = \text{diag} \left[\underbrace{\sqrt{\frac{r}{X}}, \sqrt{\frac{X}{r}}, \dots, \sqrt{\frac{r}{X}}, \sqrt{\frac{X}{r}}}_{2\ell}, \underbrace{\sqrt{\frac{X}{r}}, \sqrt{\frac{r}{X}}, \dots, \sqrt{\frac{X}{r}}, \sqrt{\frac{r}{X}}}_{2m} \right]$$

without changing its Pfaffian. That is, we may replace the kernel entries given in Theorem 5.1 with the following, without changing the correlation functions.

It is clear that this procedure changes the kernels in a manner independent of ℓ and m . The new entries of $\mathbf{K}_N^{1,1}(X; \theta, \psi)$ are given by

- $S_N^{1,1}(X; \theta, \psi) = \frac{4X^2}{\pi} \sum_{n=1}^{N/2} \frac{\cos\left(\left(n - \frac{1}{2}\right)(\theta - \psi)\right)}{(2X)^2 + (2n-1)^2}$
- $DS_N^{1,1}(X; \theta, \psi) = \frac{iXr}{\pi} \sum_{n=1}^{N/2} \frac{(2n-1) \sin\left(\left(n - \frac{1}{2}\right)(\theta - \psi)\right)}{(2X)^2 + (2n-1)^2}$
- $IS_N^{1,1}(X; \theta, \psi) = -\frac{16iX^3}{\pi r} \sum_{n=1}^{N/2} \frac{\sin\left(\left(n - \frac{1}{2}\right)(\theta - \psi)\right)}{(2n-1)((2X)^2 + (2n-1)^2)} + \frac{X}{r} \operatorname{sgn}(\psi - \theta)$

The new entries of $\mathbf{K}_N^{2,2}(X; \theta, \psi)$ are given by

- $S_N^{2,2}(X; \theta, \psi) = \frac{1}{2\pi} \sum_{n=1}^{N/2} \frac{(2n-1)^2 \cos\left(\left(n - \frac{1}{2}\right)(\theta - \psi)\right)}{(2X)^2 + (2n-1)^2}$
- $DS_N^{2,2}(X; \theta, \psi) = \frac{1}{r^2} DS_N^{1,1}(X; \theta, \psi)$
- $IS_N^{2,2}(X; \theta, \psi) = \frac{-ir}{4X\pi} \sum_{n=1}^{N/2} \frac{(2n-1)^3 \sin\left(\left(n - \frac{1}{2}\right)(\theta - \psi)\right)}{(2X)^2 + (2n-1)^2}$

The new entries of $\mathbf{K}_N^{1,2}(X; \theta, \psi)$ and $\mathbf{K}_N^{2,1}(X; \theta, \psi) = -\mathbf{K}_N^{1,2}(X; \psi, \theta)^\top$ are given by

- $S_N^{1,2}(X; \theta, \psi) = rS_N^{2,2}(X; \theta, \psi)$
- $S_N^{2,1}(X; \theta, \psi) = \frac{1}{r} S_N^{1,1}(X; \theta, \psi)$
- $DS_N^{1,2}(X; \theta, \psi) = \frac{1}{r} DS_N^{1,1}(X; \theta, \psi)$
- $IS_N^{1,2}(X; \theta, \psi) = -\frac{2}{r} DS_N^{1,1}(X; \theta, \psi)$

Computing the limit

$$\begin{aligned}
S^{1,1}(r; \theta, \psi) &= \lim_{N \rightarrow \infty} \frac{2\pi}{N} S_N^{1,1}\left(Nr; \frac{2\pi\theta}{N}, \frac{2\pi\psi}{N}\right) \\
&= \lim_{N \rightarrow \infty} 8Nr^2 \sum_{n=1}^{N/2} \frac{\cos\left(\frac{\pi}{N} (2n-1)(\theta - \psi)\right)}{(2Nr)^2 + (2n-1)^2} \\
&= 4r^2 \lim_{N \rightarrow \infty} \frac{2}{N} \sum_{n=1}^{N/2} \frac{\cos\left(\pi \left(\frac{2n-1}{N}\right)(\theta - \psi)\right)}{(2r)^2 + \left(\frac{2n-1}{N}\right)^2} \\
&= 4r^2 \int_0^1 \frac{\cos(\pi(\theta - \psi)t)}{4r^2 + t^2} dt.
\end{aligned}$$

The calculation of the analogous limits defining $DS^{1,1}$, $IS^{1,1}$, $S^{2,2}$ and $IS^{2,2}$ are all similar to that for $S^{1,1}$, and left to the reader.

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